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Abstract
Using the Verlinde formula and the symmetry of the modular matrix we describe an algorithm to find all conformal field theories with a low number of primary fields. We employ the algorithm on up to eight primary fields. Four new conformal field theories are found which do not appear to come from current algebras. This supports evidence to the fact that rational conformal field theories are far richer than previously suspected.

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Two-dimensional conformal field theory has been a source of numerous results owing to its considerable solvability, starting from the seminal works [1–8]. In these works, an axiomatic approach to conformal field theory started to emerge with the notion of conformal data.

Consider some extended algebra $\mathcal{A}$ which is the symmetry algebra of the theory. We assume that $\mathcal{A}$ contains the Virasoro algebra [1]. The Hilbert space of the theory is then some representation of this algebra, $\mathcal{H}$. The actual symmetry algebra of the theory is two copies of the algebra $\mathcal{A}$, $\mathcal{A} \times \mathcal{A}$, which act, respectively, on the left and right movers. The Hilbert space then decomposes into a sum of irreducible representations

$$\mathcal{H} = \oplus_{i,j} N_{ij} \mathcal{H}_i \times \bar{\mathcal{H}}_j, \tag{1}$$

where $\mathcal{H}_i$ denotes the irreducible representation. We shall assume that the number of irreducible representations is finite. This is called rational conformal field theory. The N_{ij} are some non-negative integers which count how many times a representation appears in the theory. The idea of conformal data is that these representations obey some relations among them which do not depend on the details of the symmetry algebra $\mathcal{A}$. These relations are common to a great deal of different algebras and form some kind of universality class among the different CFT.

We will concentrate here mainly on two main such relations, the modular transformations and the fusion rules along with the dimensions, Δ_i (mod 1) and central charge c (mod 8). Other relations exist which will not be discussed here, e.g. the braiding matrix (see, for example, [9] and references within).

The partition function of the theory is defined by the path integral on the torus with modulus τ [4, 5] and it is given by

$$Z(\tau) = \underset{\mathcal{H}}{\text{Trace}}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}, \tag{2}$$

where c is the central charge and L_0 and $\bar{L}_0$ are the left and right dimensions, and $q = e^{2\pi i \tau}$. Since $Z(\tau)$ depends only on the torus, it has to be invariant under redefinitions of the torus, called modular transformations,

$$Z(\tau + 1) = Z(-1/\tau) = Z(\tau). \tag{3}$$

We define the character of the representation $\mathcal{H}_p$ by $\chi_p = \text{Tr}_{\mathcal{H}_p} \exp[2\pi i \tau (L_0 - c/24)]$. The way this relation is obeyed is that the characters themselves form a representation of the modular group,

$$\chi_p(-1/\tau) = \sum_q S_{pq} \chi_q(\tau), \tag{4}$$

where S is some unitary matrix, along with the relation,

$$\chi_p(\tau + 1) = e^{2\pi i (\Delta_p - c/24)} \chi_p(\tau). \tag{5}$$

Then, using the characters the partition function becomes

$$Z(\tau) = \sum_{p,q} N_{p,q} \chi_p(\tau) \chi_q(\tau)^*. \tag{6}$$

The fact that S is a unitary matrix implies that we always have the, so-called, diagonal modular invariant,

$$N_{pq} = \delta_{pq}. \tag{7}$$

The modular matrix S forms part of our conformal data, where there can be many different theories with the same modular matrix.

Another important notion in the conformal data is the ring of fusion rules [1, 5, 7]. Suppose ϕ_p is the highest weight field in the p representation of the algebra. Then, we may consider what fields appear in the operator product expansion (OPE) of two such fields, $\phi_p(w)$ and $\phi_q(z)$. The OPE is then

$$\phi_p(w)\phi_q(z) = \sum_{n,t} (w - z)^{n + \Delta_t - \Delta_p - \Delta_q} [\phi_t](z), \tag{8}$$

where $[\phi_t](z)$ denotes fields in the representation $\mathcal{H}_t$ of the algebra. We thus define the fusion rules as the product in the fusion rules ring as follows:

$$[p] \times [q] = \sum_t N_{pq}^t [t], \tag{9}$$

where $[t]$ corresponds to all the fields in the OPE, equation (8), and the N_{pq}^t are non-negative integral coefficients called the fusion coefficients. Since the OPE is associative the fusion rules form a ring, called the fusion ring, where the unit is the unit representation of the chiral algebra, which we denote by '0'. The fusion ring is our second conformal data, since many theories can, in fact, have isomorphic fusion rings.

Surprisingly perhaps, and very importantly, the two notions of the modular matrix and fusion rules are related by the so-called Verlinde formula [7]. This formula gives the fusion coefficients in terms of the modular matrix,

$$N_{pq}^t = \sum_r \frac{S_{r,p} S_{r,q} S_{r,t}^\dagger}{S_{r,0}}. \tag{10}$$

Since S is unitary,

$$\sum_q S_{p,q} S_{q,r}^\dagger = \delta_{p,r}, \tag{11}$$

it follows from equation (10) that the modular matrix represents the fusion rules,

$$\psi_p \psi_q = \sum_t N_{pq}^t \psi_t, \tag{12}$$

where

$$\psi_t^{(j)} = \frac{S_{t,j}}{S_{j,0}}, \tag{13}$$

and $\psi_t^{(j)}$ is the jth eigenvalue of the fusion matrix.

Another simple, yet very powerful property of the modular matrix is its symmetry,

$$S_{p,q} = S_{q,p}, \tag{14}$$

along with its square relation,

$$(S^2)_{p,q} = b_{p,q}, \tag{15}$$

where b denotes field conjugation,

$$[\bar{p}] = \sum_q b_{pq}[q], \tag{16}$$

and b is a permutation matrix of degree 2, $b^2 = 1$. In a real theory b is the unit matrix. Otherwise it is the matrix that implements field conjugation.

Any ring which obeys the Verlinde formula with some symmetric unitary matrix S we call symmetric affine variety. From the mathematical point of view, this condition is very strong and there are very few symmetric affine varieties. In the following, we will describe an algorithm for enumerating all the symmetric affine varieties for a small number of primary fields. From our results, we believe that almost all such varieties do, in fact, correspond to the conformal data of some conformal field theory.

To check this we use an additional relation obeyed by the conformal data called Vafa equations [8]. This is a relation on the dimensions, Δ_r modulo an integer, which also form a part of our conformal data. The equations are

$$(\alpha_i \alpha_j \alpha_k \alpha_l)^{N_{ijkl}} = \prod_r \alpha_r^{N_{ijkl,r}}, \tag{17}$$

where

$$\alpha_r = \exp(2\pi i \Delta_r), \tag{18}$$

$$N_{ijkl} = \sum_m N_{ij}^{\bar{m}} N_{kl}^m, \tag{19}$$

$$N_{ijkl,r} = N_{ij}^r N_{klr} + N_{jk}^r N_{ilr} + N_{ik}^r N_{jlr}, \tag{20}$$

and we defined

$$N_{ijr} = \sum_p b_{pr} N_{ij}^p. \tag{21}$$

N_{ijr} is symmetric in all the indices. We define the matrix T as

$$T_{pq} = \delta_{pq} e^{2\pi i(\Delta_p - c/24)}, \tag{22}$$

which implements the modular transformation $\tau \rightarrow \tau + 1$, equation (5). Then the relation in the modular group implies

$$(ST)^3 = b. \tag{23}$$

This is a very strong condition on the solution of the affine variety, since there are many more equations than unknowns. Thus, we strongly believe that all the affine varieties which are solutions to Vafa equations with equation (23) obeyed, do in fact, correspond to the conformal data of some conformal field theory.

There are two further checks we can make for the consistency of the conformal data. These are based on the Galois group of the rational conformal field theory [10]. We define the level of the theory as the least integer N such that all the dimensions obey

$$N\Delta_p = \text{integer}. \tag{24}$$

We then define the discriminant as

$$D = \prod_p S_{p,0}^{-2}. \tag{25}$$

Then the discriminant is an integer, which is a product of primes which divide the level,

$$D = \prod_a p_a^{n_a}, \qquad p_a | N, \tag{26}$$

where n_a are some positive integers. For a proof, see [10]. Another property follows from the fact that the Galois group of a rational conformal field theory is always Abelian. This implies that the elements of the modular matrix has to be polynomials in some fixed root,

$$S_{ab}/S_{a,0} = P_{ab}(x), \tag{27}$$

where $x = A^{1/m}$ is some fixed root, A is rational and m is a positive integer (in all the examples it is a root of unity, $x = \exp(2\pi i/m)$) and P_{ab} are some polynomials with rational coefficients. Both properties can be checked for the symmetric affine varieties that we find, and they are indeed obeyed. This is an additional confirmation that these describe the data of some actual conformal field theories.

To conclude, we can classify all the data of rational conformal field theories (RCFT) by simply enumerating all the symmetric affine varieties which obey the additional constraints of Vafa equations and the Galois properties. These will give all the possible data for any RCFT. Unfortunately, it is very difficult to do this analytically, and we have to resort to a computer algorithm. We use this algorithm to classify all the RCFT with up to eight primary fields. The algorithm that we use is an improvement of that already used in [11], where RCFT with up to six primary fields were studied.

The idea is to generate all the fusion rings up to some limitation on the structure constant

$$N_{ijk} \leqslant M, \tag{28}$$

where M is some positive integer. Then, we diagonalize the fusion ring to get the modular matrix, using Verlinde formula (equation (10)). Once we get the modular matrix, we check if it is symmetric. This gives all the symmetric affine varieties.

To be more specific, denote by '0' the unit primary field. We choose one special field different from the unity, denoted by '1'. We can then generate all the fusion coefficients of the type $N_{1,i,j}$ which can be any symmetric matrix with non-negative integer coefficients, with the only constraint that $N_{1,0}^j = \delta_{j,1}$. In practice, we have to limit the fusion coefficients up to some value M, equation (28). Here we take M to be equal to 1. This limits our search to fusion rings where at least one field has fusion coefficients not exceeding one.

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Now, once a candidate fusion ring is generated we can use

$$\psi_1 \psi_j = \sum_k N_{1j}^k \psi_k, \tag{29}$$

where ψ_i was defined in equation (13). This means that the modular matrix is the normalized eigenmatrix, denoted by Q, of the matrix N_{1i}^j. We still, however have the freedom of permuting the columns of the eigenmatrix and the freedom of multiplying each column by ± 1. Only after permuting the columns with all possible permutations, and multiplying by all the possible signs, we can check if the resulting matrix is symmetric. Here, however, we improve the algorithm by using a 'pre-filtration' of the ring. We compare the first row of Q to each of the columns of Q, and if the vectors are not the same (while ordered and up to signs) we can discard the ring. Only if the ring passes the pre-filtration we continue with generating all the permutations and signs. This simple improvement accelerates considerably the speed of execution in a computer program, allowing us to study all rings with up to eight primary fields.

There are two complications to be dealt with. First, if some of the eigenvalues of the N_{1i}^j, denoted by λ_j are degenerate, say $\lambda_s = \lambda_t$, the matrix of eigenvectors Q is not uniquely determined, as we can have any linear combination of the s and t columns. We avoid the problem by checking the first column (different from s and t) against any row, while allowing the s and t columns to be some unknown numbers. This provides pre-filtration for the degenerate case. If the ring passes this pre-filtration, we check for the symmetry of the full matrix (with unknown s and t columns), and then continue by generating another fusion coefficient N_{2i}^j, constraining associativity, thus removing the degeneracy.

The second complication is if the ring is not real, i.e., we have some non-trivial field conjugation matrix, b. This we deal with by generating all the possible values of b and checking for the symmetry of the modular matrix for each b.

Once a candidate fusion variety has been found, we proceed to solve Vafa equations (17) and $(ST)^3 = b$ to get the dimensions and central charges. This way we classify all the RCFT with up to eight primary fields. Most of the data can be seen to be not new, belonging to some current algebra model which is already known. We do find however four new fusion rings which do not seem to correspond to any current algebra. We have two new models with six primary fields and two with seven. The conformal data of these models will be listed below. We believe that these correspond, in fact, to some full-fledged rational conformal field theories, yet to be explored.

If this conjecture is correct, this is a very important find. To this date all explicitly known RCFT could be built from current algebras, i.e., WZW models with various constructions. This is to the extent that a widely held belief by workers in the field was that these are the only existing RCFTs. Our results show that this is not the case and that the realm of RCFT is indeed very rich.

(1) Conformal field theory A_6 with six primary fields. The fusion rules are given by

$$\phi_1^2 = 1 + \phi_2 + \phi_3$$
$$\phi_2^2 = 1 + \phi_3 + \phi_5$$
$$\phi_3^2 = 1 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_4^2 = 1 + \phi_1 + \phi_2 + 2\phi_3 + 2\phi_4 + \phi_5$$
$$\phi_5^2 = 1 + \phi_1 + \phi_3$$
$$\phi_1\phi_2 = \phi_1 + \phi_4$$
$$\phi_1\phi_3 = \phi_1 + \phi_3 + \phi_4$$
$$\phi_1\phi_4 = \phi_2 + \phi_3 + \phi_4 + \phi_5$$

$$\phi_1\phi_5 = \phi_4 + \phi_5$$
$$\phi_2\phi_3 = \phi_2 + \phi_3 + \phi_4$$
$$\phi_2\phi_4 = \phi_1 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_2\phi_5 = \phi_2 + \phi_4$$
$$\phi_3\phi_4 = \phi_1 + \phi_2 + \phi_3 + 2\phi_4 + \phi_5$$
$$\phi_3\phi_5 = \phi_3 + \phi_4 + \phi_5$$
$$\phi_4\phi_5 = \phi_1 + \phi_2 + \phi_3 + \phi_4. \tag{30}$$

Solving Vafa equations, equation (17), we get the dimensions of the fields which are, in the order $\phi_0 = 1, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ given by $0, -4/9, -1/9, 1/3, -1/3, 2/9$ modulo an integer. The central charge, which solves equation (23) is given by $8/3$ modulo 4. The discriminant, equation (25), is given by 3^{12}. Note that this is just one of the solutions to Vafa equations due to the fact that we can permute some of the rows of the modular matrix, leaving it symmetric, a phenomenon called pseudo-conformal field theory. Other solutions will be just a multiplication by an integer factor s of the dimensions and central charge, where s has to be strange to the level, N, equation (24), which is given here by 9.

(2) Conformal field theory B_6 with six primary fields. Here the fusion rules are given by

$$\phi_1^2 = 1 + \phi_1 + \phi_2 + \phi_4$$
$$\phi_2^2 = 1 + \phi_2 + \phi_4 + \phi_5$$
$$\phi_3^2 = 1 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_4^2 = 1 + \phi_1 + \phi_2 + 2\phi_3 + 2\phi_4 + \phi_5$$
$$\phi_5^2 = 1 + \phi_1 + \phi_4 + \phi_5$$
$$\phi_1\phi_2 = \phi_1 + \phi_3 + \phi_4$$
$$\phi_1\phi_3 = \phi_2 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_1\phi_4 = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 \tag{31}$$
$$\phi_1\phi_5 = \phi_3 + \phi_4 + \phi_5$$
$$\phi_2\phi_3 = \phi_1 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_2\phi_4 = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_2\phi_5 = \phi_2 + \phi_3 + \phi_4$$
$$\phi_3\phi_4 = \phi_1 + \phi_2 + \phi_3 + 2\phi_4 + \phi_5$$
$$\phi_3\phi_5 = \phi_1 + \phi_2 + \phi_3 + \phi_4$$
$$\phi_4\phi_5 = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5.$$

The dimensions are given by $0, 2/7, -3/7, 0, 1/3, 1/7$ modulo an integer, and the central charge is -2 modulo 4. The discriminant is $D = 3^3 7^5$.

(3) Conformal field theory A_7 with seven primary fields. The fusion rules are

$$\phi_1^2 = 1$$
$$\phi_2^2 = 1 + \phi_1 + \phi_5$$
$$\phi_3^2 = 1 + \phi_2 + \phi_5 + \phi_6$$
$$\phi_4^2 = 1 + \phi_2 + \phi_5 + \phi_6$$
$$\phi_5^2 = 1 + \phi_1 + \phi_6$$

$$\phi_6^2 = 1 + \phi_1 + \phi_2$$
$$\phi_1\phi_2 = \phi_2$$
$$\phi_1\phi_3 = \phi_4$$
$$\phi_1\phi_4 = \phi_3$$
$$\phi_1\phi_5 = \phi_5$$
$$\phi_1\phi_6 = \phi_6$$
$$\phi_2\phi_3 = \phi_3 + \phi_4$$
$$\phi_2\phi_4 = \phi_3 + \phi_4$$
$$\phi_2\phi_5 = \phi_2 + \phi_6$$
$$\phi_2\phi_6 = \phi_5 + \phi_6$$
$$\phi_3\phi_4 = \phi_1 + \phi_2 + \phi_5 + \phi_6$$
$$\phi_3\phi_5 = \phi_3 + \phi_4$$
$$\phi_3\phi_6 = \phi_3 + \phi_4$$
$$\phi_4\phi_5 = \phi_3 + \phi_4$$
$$\phi_4\phi_6 = \phi_3 + \phi_4$$
$$\phi_5\phi_6 = \phi_2 + \phi_5. \tag{32}$$

The dimensions are here $0, 0, -1/7, 3/8, -1/8, 3/7, -2/7$ modulo an integer, and the central charge is -2 modulo 4. The discriminant is $D = 2^8 7^5$.

(4) The conformal field theory B_7 with seven primary fields. The fusion rules are

$$\phi_1^2 = \phi_3 + \phi_5$$
$$\phi_2^2 = \phi_3 + \phi_5$$
$$\phi_3^2 = 1 + \phi_3 + \phi_4$$
$$\phi_4^2 = 1 + \phi_3 + \phi_4$$
$$\phi_5^2 = 1$$
$$\phi_6^2 = 1 + \phi_3 + \phi_4 + \phi_5$$
$$\phi_1\phi_2 = 1 + \phi_4$$
$$\phi_1\phi_3 = \phi_2 + \phi_6$$
$$\phi_1\phi_4 = \phi_1 + \phi_6$$
$$\phi_1\phi_5 = \phi_2$$
$$\phi_1\phi_6 = \phi_3 + \phi_4$$
$$\phi_2\phi_3 = \phi_1 + \phi_6$$
$$\phi_2\phi_4 = \phi_2 + \phi_6$$
$$\phi_2\phi_5 = \phi_1$$
$$\phi_2\phi_6 = \phi_3 + \phi_4$$
$$\phi_3\phi_4 = \phi_3 + \phi_4 + \phi_5$$
$$\phi_3\phi_5 = \phi_4$$
$$\phi_3\phi_6 = \phi_1 + \phi_2 + \phi_6$$
$$\phi_4\phi_5 = \phi_3$$
$$\phi_4\phi_6 = \phi_1 + \phi_2 + \phi_6$$
$$\phi_5\phi_6 = \phi_6.$$

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The dimensions are $0, 1/32, 1/32, 1/4, -1/4, 1/2, -11/32$, and the central charge is $11/4$. The discriminant is given by $D = 2^{22}$.

The rest of the conformal field theories data, apart from the four new ones above, can be seen to come from current algebras, i.e., they are either WZW models or quotients of such models. This completes the classification of all conformal field theories with up to eight primary fields. However, due to large running time there may be additional degenerate cases for eight primary fields, which we did not analyze completely.

Naturally, the important question of how to construct explicitly these four conjectured conformal field theories arise. We know only the data for these RCFTs. We can however proceed along the lines of [12]. First, we find the braiding matrix by solving the hexagon relation (e.g. [9]). Then we find a lattice model by 'Baxterising' the braiding matrix along the lines described in [12]. This will give a solvable IRF model with the admissibility condition given by the appropriate fusion rules (see [12] for more details). Calculating the fixed point RCFT of this lattice model will then give an RCFT related to the original RCFT (in the appropriate regime of the model). Work along these lines is currently in progress [13], and we hope to report on it elsewhere.

We hope that this work will be the first step in exploring the vast richness of rational conformal field theories. As was demonstrated here, current algebras are just a small subset of allowable conformal data. It is left to the future work to explicitly build on these theories.

Acknowledgments

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